On the Density of Transitive Tournaments

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Abstract

We prove that for every fixed k, the number of occurrences of the transitive tournament Tr_k of order k in a tournament T_n on n vertices is asymptotically minimized when T_n is random. In the opposite direction, we show that any sequence of tournaments $\{T_n\}$ achieving this minimum for any fixed $k \geqslant 4$ is necessarily quasi-random. We present several other characterizations of quasi-random tournaments nicely complementing previously known results and relatively easily following from our proof techniques.

For a fixed combinatorial object M, e.g. a graph, a hypergraph or a tournament, the number of its occurrences in a large random object N of the same signature is always easy to compute asymptotically. Reverse problems have become the subject of active research in combinatorics.

In the first direction along these lines, we are interested in the questions of the kind "for which templates M the number of occurrences is asymptotically optimized by the random object" or, in other terms, when the latter forms an extremal configuration for the corresponding extremal problem. Perhaps, one of the most prominent settings in which this question has been studied systematically is that of *k-common graphs* [JŠT96]. In

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this setting, M is an ordinary graph, N is a k-edge-coloring of a complete graph, and an "occurrence" means a monochromatic copy. The k-common graphs are those for which the random edge coloring asymptotically minimizes the number of occurrences, and they were studied in many papers, see e.g. [Erd62, BR80, Sid89, JŠT96, Tho89, CY11, HHK$^+$12].

The second direction focusses on the complementary question: "When all extremal configurations for this extremal problem are necessarily quasi-random?" We do not attempt to review here the theory of quasi-randomness: beginning from the seminal papers by Thomason [Tho87] and Chung, Graham, Wilson [CGW89], it has developed into a vast field surveyed for example in [KS06]. But the main thrust of this theory is that many a priori different properties eventually lead to the same class of combinatorial objects, hereafter called *quasi-random*. Perhaps, the most intuitive of these properties says that quasi-random objects are precisely those N in which *all* fixed templates M have the right density of occurrences, understood asymptotically.

In this note we study questions of both kinds for tournaments. By analogy with graphs, it might be tempting to call a tournament T *common* if for any increasing sequence $\{T_n\}$ of tournaments we have

$$\liminf_{n\to\infty} p(T, T_n) \geqslant \frac{k!}{|\operatorname{Aut}(T)| \cdot 2^{\binom{k}{2}}}, \tag{1}$$

where $k \overset{\text{def}}{=} |V(T)|$, $p(T, T_n)$ is the density with which T appears in T_n as an unlabelled sub-tournament and $\operatorname{Aut}(T)$ denotes the group of automorphisms of T. In the language of flag algebras [Raz07] it can be stated more cleanly and concisely as

$$\forall \phi \in \operatorname{Hom}^+(\mathcal{A}^0[T_{\text{Tournaments}}], \mathbb{R}) \left(\phi(T) \geqslant \frac{k!}{|\operatorname{Aut}(T)| \cdot 2^{\binom{k}{2}}} \right), \tag{2}$$

and in what follows we will mostly use this language. The reader who feels uncomfortable with limit objects should have little difficulties translating our statements to the finite world by analogy with (1).

Upon a moment's reflection it becomes clear that only transitive tournaments may possibly be common since all others occur with zero density in the increasing sequence $\{\operatorname{Tr}_n\}$ of transitive tournaments. This gives rise to the natural question if the converse is true, that is if

$$\phi(\operatorname{Tr}_k) \geqslant \frac{k!}{2^{\binom{k}{2}}}, \tag{3}$$

where again ϕ is any algebra homomorphism from $\mathrm{Hom}^+(\mathcal{A}^0[T_{\mathrm{Tournaments}}], \mathbb{R})$. To the best of our knowledge, this question was not explicitly asked before. The cases $k = 0, 1, 2$ are trivial, and the case $k = 3$ was implicitly answered by Chung and Graham in [CG91, Fact 1]. With a bit of effort, the bound (3) for $k = 4$ can be extracted from Griffith's paper [Gri13, Proposition 3.1 (i)], but his proof does not seem to be generalizable to larger values of k.

Our first main result confirms (3) for all values of k. In other words, transitive tournaments are common and vice versa. This result also has an interesting "sharp threshold" flavor to it. Namely, if we take an arbitrarily large transitive tournament and flip even a *single* arc in it[1], the extremal (minimizing) configuration jumps from the random tournament to the opposite side of the spectrum, i.e. transitive tournaments.

The theory of quasi-random tournaments was inaugurated by Chung and Graham in [CG91] and followed up in [KS13, Gri13]. A significant portion of all these papers is devoted to the question formulated above in more general context: for which tournaments T the equality in (2) implies that ϕ is quasi-random? We show this for any transitive tournament Tr_k with[2] $k \geqslant 5$.

Many characterizations in [CG91, Gri13] are given in terms of "flag concentration". Given a tournament T and an edge $\langle u, v \rangle \in E(T)$, all other vertices $w \in V(T) \setminus \{u, v\}$ can be classified into four classes ("flags"):

1. $\langle u, w \rangle, \langle v, w \rangle \in E(T)$,

2. $\langle w, u \rangle, \langle w, v \rangle \in E(T)$,

3. $\langle u, w \rangle, \langle w, v \rangle \in E(T)$.

4. $\langle v, w \rangle, \langle w, u \rangle \in E(T)$,

Following and expanding a bit the notation in [Raz13a], we let $O^A(u, v)$, $I^A(u, v)$, $\mathrm{Tr}_3^A(u, v)$ and $\vec{C}_3^A(u, v)$ denote the numbers of vertices in the four classes (taken in this order, see also Figure 1 below) divided by $|V(T)| - 2$. One of the keystone characterizations in [CG91] (P_4, to be exact) says that a sequence of tournaments is quasi-random if and only if[3] $O^A(u, v)$ is

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[1]except for the arcs that when flipped yield transitive tournaments again

[2]The case $k = 4$ can be again extracted from [Gri13].

[3]This again can be more cleanly stated in the language of flag algebras [Raz07, §3.2] (which we do in Section 1) or digraphons [DJ08, Example 9.2].

"nearly" 1/4 for "almost all" edges $\langle u, v \rangle$. A similar statement for $I^A(u, v)$ follows by duality.

Our methods allow us to do, in exactly the same manner, the two remaining cases: $\mathrm{Tr}_3^A(u, v)$ and $\vec{C}_3^A(u, v)$. Stated in the finite language and combined with the previously known results, we now have that a sequence of tournaments $\{T_n\}$ is quasi-random if and only if

$$\sum_{\langle u,v \rangle \in E(T_n)} |F(u, v) - 1/4| \leqslant o(n^2), \tag{4}$$

where F corresponds to *any* of the four cases above.

The note is organized as follows. In Section 1 we remind some rudimentary concepts from the theory of flag algebras and show how to treat quasi-randomness in that context. Again, the reader who feels uncomfortable with this language (and is willing to tolerate a bit of coping with low-order error terms instead) should have no difficulties replacing σ-extensions with averaging over vertices or arcs in large but finite tournaments, see e.g. [Raz13b]. In Section 2 we prove the bound (2), and also that the equality is attained here if and only if ϕ is quasi-random (Theorem 2.1). Finally, in Section 3 we prove additional characterizations of quasi-random tournaments in terms of "flag concentrations" (Theorems 3.1 and 3.2).

1   Quasi-Randomness in Flag Algebras

In this section, we translate the results of quasi-randomness that we are going to use to the language of flag algebras. We assume the reader has some familiarity with not only the basic setting of flag algebra, but also with extensions of homomorphisms [Raz07, §3.2].

Following the notation of [Raz07, Raz13a], we consider the theory of tournaments $T_{\mathrm{Tournaments}}$ and we let 1 denote the (unique) type of size 1 and A denote the type of size 2 such that the vertex labelled with 1 beats the other (labelled) vertex (see Figure 1). For a type σ, we denote the unity of the algebra $\mathcal{A}^\sigma$ by 1_σ, and, as always, 1_0 is abbreviated to 1.

We have already introduced the notation Tr_k to denote the transitive tournament of size k. We let $\vec{C}_3$ be the 3-directed cycle (that is, the only other tournament of size 3) and we define the following tournaments of size 4.

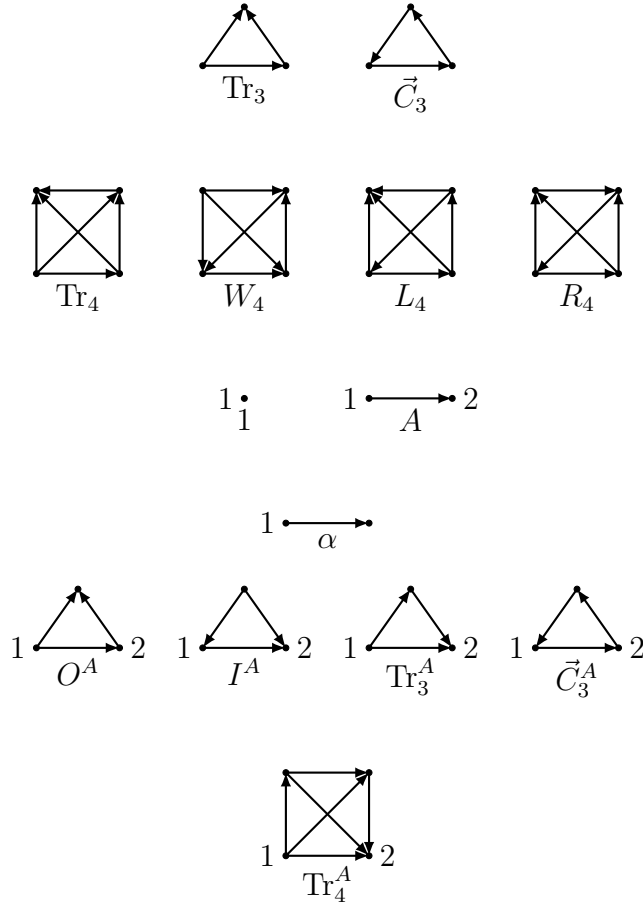1. The tournament W_4, which is the (unique) *non-transitive* tournament

Figure 1: Types and flags used.

of size 4 that has a vertex with outdegree 3 (that is, there is a "winner" in W_4);

2. The tournament L_4, which is the (unique) *non-transitive* tournament of size 4 that has a vertex with indegree 3 (that is, there is a "loser" in L_4);

3. The tournament R_4, which is the (unique) tournament of size 4 that has outdegree sequence $(1, 1, 2, 2)$.

Note that along with Tr_4, this list covers all tournaments of size 4.

We define the 1-flag α as the (unique) 1-flag of size 2 in which the labelled vertex beats the unlabelled vertex. We also define the following A-flags of size 3.

1. The flag O^A, in which the only unlabelled vertex is beaten by both labelled vertices;

2. The flag I^A, in which the only unlabelled vertex beats both labelled vertices;

3. The flag Tr_3^A, which is the only remaining A-flag whose underlying model is Tr_3;

4. The flag $\vec{C}_3^A$, which is the only A-flag whose underlying model is $\vec{C}_3$.

Again, this is the complete list of A-flags of size 3.

We also follow the original notation of flag algebras when using the downward operator $[\![\,\cdot\,]\!]_\sigma$ to the 0-algebra or when using σ-extensions of homomorphisms $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ (which are denoted by ϕ^σ). We remind that ϕ^σ can be conveniently viewed [Raz07, Definition 10] as the unique $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$-valued random variables satisfying the identities

$$\mathbf{E}[\phi^\sigma(F)] = \frac{\phi([\![F]\!]_\sigma)}{\phi([\![1_\sigma]\!]_\sigma)} \tag{5}$$

for every $F \in \mathcal{F}^\sigma$.

Finally, for a flag F_0 of type σ with $|\sigma| + 1$ vertices we have the algebra homomorphism $\pi^{F_0}\colon \mathcal{A}^0 \longrightarrow \mathcal{A}^\sigma_{F_0}$, where $\mathcal{A}^\sigma_{F_0}$ is the localization of the algebra $\mathcal{A}^\sigma$ with respect to the multiplicative system $\{F_0^\ell \,|\, \ell \in \mathbb{N}\}$ [Raz07, §2.3.2]. Intuitively, it corresponds to taking the sub-model induced by the flag F_0, and the localization is needed for proper normalization resulting from decreasing the set of vertices. In this note we will only need the operator π^{O^A}.

We will not need these concepts in the more complicated scenario when the smaller type is also non-trivial.

Let us denote the homomorphism of $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ corresponding to the random tournament by ϕ_{qr}, that is, it is the almost sure limit of the sequence of random tournaments (where each arc orientation is picked independently with probability $1/2$) when the number of vertices goes to infinity.

Note that for every tournament T, we have

$$\phi_{\mathrm{qr}}(T) = \frac{|V(T)|!}{|\mathrm{Aut}(T)| \cdot 2^{\binom{|T|}{2}}}.$$

In particular, since transitive tournaments possess only trivial automorphisms, we have

$$\phi_{\mathrm{qr}}(\mathrm{Tr}_k) = \frac{k!}{2^{\binom{k}{2}}},$$

for every $k \in \mathbb{N}$.

As for the quasi-randomness properties (the P properties of Chung–Graham [CG91]), we are interested in the following ones.

- P_2: $\phi(\mathrm{Tr}_4 + R_4) = 3/4$;

- P_4: $\phi^{\boldsymbol{A}}(O^A) = 1/4$ a.s.

That is, the above are equivalent to each other, and are equivalent to the fact $\phi = \phi_{\mathrm{qr}}$ (which, in the terminology of [CG91], is precisely $\forall s \in \mathbb{N}, P_1(s)$).

The translation of property P_2 follows from simple arithmetics relating homomorphism densities to flag algebra densities, and the translation of P_4 follows from the definition of extensions of homomorphisms.

2 The Minimum Density of Transitive Tournaments

We start by defining for every $k \geqslant 2$ the 1-flag Tr_k^W as the flag obtained from Tr_k by labelling its winner (i.e. the unique vertex with outdegree $k-1$) and the A-flag Tr_k^{W2} as the flag obtained from Tr_k by labelling its winner with the label 1 and its runner-up (i.e. the unique vertex with outdegree $k-2$) with the label 2. In particular, with this definition we have $\alpha = \mathrm{Tr}_2^W$ and $O^A = \mathrm{Tr}_3^{W2}$.

We further note that $\alpha^2 = \mathrm{Tr}_3^W$, $[\![\alpha]\!]_1 = \mathrm{Tr}_2 / 2 = 1/2$ and for every $k \geqslant 2$, we have

$$\mathrm{Tr}_k = k[\![\mathrm{Tr}_k^W]\!]_1 = k(k-1)[\![\mathrm{Tr}_k^{W2}]\!]_A;$$
$$\mathrm{Tr}_k^{W2} = \pi^{O^A}(\mathrm{Tr}_{k-2}) \cdot (O^A)^{k-2}.$$

We are now ready to prove the main result. We remind the reader that, although the proof is presented for all $k \in \mathbb{N}$, the result for $k \leqslant 4$ was known before.

Theorem 2.1. *In the theory of tournaments, for every $k \in \mathbb{N}$ and every $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, we have*

$$\phi(\mathrm{Tr}_k) \geqslant \frac{k!}{2^{\binom{k}{2}}}.$$

Furthermore, for any fixed $k \geqslant 4$, the equality holds if and only if ϕ is the quasi-random homomorphism ϕ_{qr}.

Proof. For $k = 0, 1, 2$, the result is trivial, because as flag algebra elements, we have $\mathrm{Tr}_0 = \mathrm{Tr}_1 = \mathrm{Tr}_2 = 1$ and the right hand side of the formula evaluates to 1 in these cases.

For $k = 3$, note that since $[\![1_1]\!]_1 = 1$, we have

$$\phi(\mathrm{Tr}_3) = 3\phi([\![\alpha^2]\!]_1) = 3\mathbf{E}\big[\boldsymbol{\phi^1}(\alpha)^2\big] \geqslant 3\mathbf{E}\big[\boldsymbol{\phi^1}(\alpha)\big]^2 = 3\phi([\![\alpha]\!]_1)^2 = \frac{3}{4} = \frac{3!}{2^{\binom{3}{2}}},$$

Now we proceed by induction. Suppose that $k \geqslant 4$ and that the result is valid for $k - 2$.

Since $[\![1_A]\!]_A = 1/2$, we have

$$\phi(\mathrm{Tr}_k) = k(k-1)\phi([\![\pi^{O^A}(\mathrm{Tr}_{k-2}) \cdot (O^A)^{k-2}]\!]_A)$$
$$= \frac{k(k-1)}{2}\mathbf{E}\Big[\boldsymbol{\phi^A}(\pi^{O^A}(\mathrm{Tr}_{k-2}))\boldsymbol{\phi^A}\big((O^A)^{k-2}\big)\Big].$$

Since $\boldsymbol{\phi^A}(O^A) > 0$ implies that $\phi^A \circ \pi^{O^A} \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, by inductive hypothesis we have

$$\mathbf{E}\Big[\boldsymbol{\phi^A}(\pi^{O^A}(\mathrm{Tr}_{k-2}))\boldsymbol{\phi^A}\big((O^A)^{k-2}\big)\Big] \geqslant \frac{(k-2)!}{2^{\binom{k-2}{2}}}\mathbf{E}\big[\boldsymbol{\phi^A}((O^A)^{k-2})\big].$$

By Jensen's inequality, we have

$$\mathbf{E}\big[\boldsymbol{\phi^A}(O^A)^{k-2}\big] \geqslant \mathbf{E}\big[\boldsymbol{\phi^A}(O^A)\big]^{k-2} = (2\phi([\![O^A]\!]_A))^{k-2} = \left(\frac{\phi(\mathrm{Tr}_3)}{3}\right)^{k-2}. \quad (6)$$

Since we already proved the case $k = 3$, we have $\phi(\mathrm{Tr}_3) \geqslant 3/4$, so putting things together, we get

$$\phi(\mathrm{Tr}_k) \geqslant \frac{k(k-1)}{2} \cdot \frac{(k-2)!}{2^{\binom{k-2}{2}}} \left(\frac{\phi(\mathrm{Tr}_3)}{3}\right)^{k-2} \geqslant \frac{k!}{2^{\binom{k}{2}}}. \tag{7}$$

For the "furthermore" part, note that if equality holds for ϕ, then along the proof we have equalities instead of inequalities.

In particular, since Jensen's inequality was used in (6) with the function x^{k-2}, which for $k \geqslant 4$ is strictly convex[4] on $[0, +\infty)$, we have that $\phi^{\boldsymbol{A}}(O^A)$ is a.s. a constant, denote it by C. Furthermore, we also have $\phi(\mathrm{Tr}_3) = 3/4$ as it is used in the chain of inequalities (7). This allows us to compute C as follows:

$$C = \mathbf{E}\big[\phi^{\boldsymbol{A}}(O^A)\big] = 2\phi(\llbracket O^A \rrbracket_A) = \frac{\phi(\mathrm{Tr}_3)}{3} = \frac{1}{4}.$$

Therefore ϕ satisfies P_4, hence $\phi = \phi_{\mathrm{qr}}$. ∎

Remark. *Note that if we wanted to prove just the inequality statement (i.e. without the "furthermore" part), we could have done the induction with a simpler argument involving $\pi^\alpha(\mathrm{Tr}_{k-1})\alpha^{k-1}$ instead of $\pi^{O^A}(\mathrm{Tr}_{k-2}) \cdot (O^A)^{k-2}$. Moreover, modulo the following (straightforward) generalization of Cauchy–Schwarz inequality [Raz07, (22)]:*

$$\llbracket f^k \rrbracket_\sigma \cdot \llbracket 1_\sigma \rrbracket_\sigma^{k-1} \geqslant \llbracket f \rrbracket_\sigma^k$$

we could have also avoided the extension $\phi^{\boldsymbol{A}}$ at all and reduced the whole argument to a several-lines calculation.

3 New Characterizations with A-Flags and A-Extensions

As we already said in the introduction, Chung and Graham presented in [CG91] quasi-random characterizations involving the A-flag O^A. Reverting all arcs of the tournament, we get the dual characterization involving I^A (which is stated below for completeness).[5]

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[4]This is precisely the reason why the second part of Theorem 2.1 does not hold for $k = 3$.

[5]Just reverting the arcs, we actually get a characterization involving flags of the other type of size 2, but then we may use the flag algebra isomorphism that swaps the labels 1 and 2 to arrive at the desired characterization.

- P_4': $\phi^{\boldsymbol{A}}(I^A) = 1/4$ a.s.

In this section, we prove that the two remaining A-flags Tr_3^A and $\vec{C}_3^A$ of size 3 also characterize quasi-randomness of tournaments. Note, however, that these flags are self-dual, so they must be treated separately.

Theorem 3.1. *In the theory of tournaments, for every $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, we have that*

$$\phi^{\boldsymbol{A}}(\mathrm{Tr}_3^A) = \frac{1}{4} \quad \text{a.s.}$$

if and only if ϕ is the quasi-random homomorphism ϕ_{qr}.

Proof. This is similar to the second part of the proof of Theorem 2.1, except that now we use the flag Tr_3^A instead of O^A. Let Tr_4^A denote the A-flag obtained from Tr_4 by labelling its winner with label 1 and labelling its loser with label 2 (see Figure 1).

Note that $(\mathrm{Tr}_3^A)^2 = \mathrm{Tr}_4^A$ and that $\mathrm{Tr}_4 = 12 \left[\!\left[\mathrm{Tr}_4^A \right]\!\right]_A$. Since $\left[\!\left[1_A \right]\!\right]_A = 1/2$, we have

$$\phi(\mathrm{Tr}_4) = 12\phi(\left[\!\left[(\mathrm{Tr}_3^A)^2 \right]\!\right]_A) = 6\mathbf{E}\left[\phi^{\boldsymbol{A}}(\mathrm{Tr}_3^A)^2\right] \geqslant 6\mathbf{E}\left[\phi^{\boldsymbol{A}}(\mathrm{Tr}_3^A)\right]^2 = 6(2\phi(\left[\!\left[\mathrm{Tr}_3^A \right]\!\right]_A))^2 = \frac{2}{3}\phi(\mathrm{Tr}_3)^2.$$

By Theorem 2.1, we know that $\phi(\mathrm{Tr}_3) \geqslant 3/4$, hence we have

$$\phi(\mathrm{Tr}_4) \geqslant \frac{3}{8} = \frac{4!}{2^{\binom{4}{2}}}.$$

But, since x^2 is both strictly convex and strictly increasing on $[0, +\infty)$, we have that equality holds if and only if we have almost surely

$$\phi^{\boldsymbol{A}}(\mathrm{Tr}_3^A) = \mathbf{E}\left[\phi^A(\mathrm{Tr}_3^A)\right] = \frac{\phi(\mathrm{Tr}_3)}{3} = \frac{1}{4}.$$

By the "furthermore" part of Theorem 2.1, we have

$$\phi^{\boldsymbol{A}}(\mathrm{Tr}_3^A) = \frac{1}{4} \quad \text{a.s.} \iff \phi(\mathrm{Tr}_4) = \frac{4!}{2^{\binom{4}{2}}} \iff \phi = \phi_{qr}. \qquad \blacksquare$$

For the final flag $\vec{C}_3^A$ the proof cannot be done analogously because the underlying tournament is now $\vec{C}_3$ instead of Tr_3. Nevertheless, we are able to obtain the result by other means.

Theorem 3.2. *In the theory of tournaments, for every $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$, we have that*

$$\phi^{\boldsymbol{A}}(\vec{C}_3^A) = \frac{1}{4} \quad \text{a.s.}$$

if and only if ϕ is the quasi-random homomorphism ϕ_{qr}.

Proof. In one direction it is simple. By Chung and Graham's characterizations P_4 and P_4' (that is, after reverting the arcs) and by Theorem 3.1, we have

$$\phi_{\mathrm{qr}}^{\boldsymbol{A}}(O^A) = \phi_{\mathrm{qr}}^{\boldsymbol{A}}(I^A) = \phi_{\mathrm{qr}}^{\boldsymbol{A}}(\mathrm{Tr}_3^A) = \frac{1}{4} \quad \text{a.s.}$$

Hence, since $\vec{C}_3^A = 1_A - O^A - I^A - \mathrm{Tr}_3^A$, we have

$$\phi_{\mathrm{qr}}^{\boldsymbol{A}}(\vec{C}_3^A) = \frac{1}{4} \quad \text{a.s.}$$

For the converse, note first that if $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$ is such that $\phi^{\boldsymbol{A}}(\vec{C}_3^A) = 1/4$ a.s., then

$$\phi(\vec{C}_3) = \frac{\phi(\llbracket \vec{C}_3^A \rrbracket_A)}{\phi(\llbracket 1_A \rrbracket_A)} = \mathbf{E}\left[\phi^{\boldsymbol{A}}(\vec{C}_3^A)\right] = \frac{1}{4}.$$

Now, it is straightforward to check the following flag algebra identities.

$$R_4 = 12\llbracket (\vec{C}_3^A)^2 \rrbracket_A; \qquad\qquad \vec{C}_3 = \frac{1}{2}R_4 + \frac{1}{4}W_4 + \frac{1}{4}L_4.$$

Using the first identity along with $\phi^{\boldsymbol{A}}(\vec{C}_3^A) = 1/4$ a.s., we get

$$\phi(R_4) = 12\phi(\llbracket (\vec{C}_3^A)^2 \rrbracket_A) = 6\mathbf{E}\left[\phi^{\boldsymbol{A}}(\vec{C}_3^A)^2\right] = \frac{3}{8}.$$

Using the second identity, we get

$$\phi(W_4 + L_4) = 4\phi(\vec{C}_3) - 2\phi(R_4) = \frac{1}{4},$$

hence $\phi(\mathrm{Tr}_4 + R_4) = 3/4$ (since $\mathrm{Tr}_4 + R_4 + W_4 + L_4 = 1$), which is property P_2 of Chung–Graham.

Therefore $\phi = \phi_{\mathrm{qr}}$. ∎

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